Retarded Integration in Classical Electrodynamics

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The bound and emitted four-momentum of an accelerated point charge is calculated in a direct manner without use of Gauß' or Stokes' integral theorems. This new integration procedure accounts most naturally for the retarded character of the electromagnetic interactions and, if applied to the bound four-momentum, reveals that the asymptotic condition of uniform motion in the distant past can be weakened essentially.

I. Introduction

The recent progress in single particle electrodynamics is mainly due to the circumstance that one has learned to compute the particle's energy-momentum (and angular momentum) as an integral of the retarded field over some hypersurface in space-time [1]. Once the four-momentum carried by the field of the particle is calculated, one can easily derive its equation of motion by evoking the conservation laws, either in differential or in global form [2–5]. The equation with which one ends up in such a procedure is the well-known Lorentz-Dirac equation [6, 7].

Though there are some difficulties with the latter equation [8], we restrict ourselves in the present paper to certain problems occuring in connection with the calculation of the particle four-momenta. It has turned out that the integration process mentioned above does not lead to a unique result, unless one imposes the restriction of uniform motion in the distant past. We first re-examine the calculations of the particle momentum of various authors with respect to that asymptotic condition, and we shall find that the asymptotic condition is merely the consequence of an "unphysical" choice of the boundary of the integration region. If the boundary is chosen in agreement with the physical process of retarded emision of the electromagnetic actions from the point source, one can actually dispense with the above mentioned asymptotic condition and one finds, instead, a much weaker one to be imposed on the particle motion in the distant past [cf. condition (III,16)]. For instance, whenever the particle four-velocity is bounded in the past, the bound four-momentum can be defined uniquely. The weakened condition can be applied to all methods, which are used in literature to circumvent a direct integration of field momentum by resorting to Gauß' or Stokes' theorems and computing the integral over some auxiliary surface.

However, the retarded integration procedure, which follows closely the physical mechanism of retarded light emission, is brought out more clearly by the direct integration method presented below, where no auxiliary surfaces or integral theorems are to be used.

II. Boundary-dependent Surface Integrals

In the following, we want to deal with a point charge, the field of which is given by the well-known Liénard-Wiechert potentials

$$A^{\lambda}(x) = Z \cdot u^{\lambda}/R\,,$$

where $\{u^{\lambda}\}$ is the four-velocity, normalized to unity $u^{\lambda}u_{\lambda}=1$, Z designates the charge of the particle, and R is the retarded distance between field point $\{x^{\nu}\}$ and source point $\{z^{\lambda}(s)\}$ on the particle world line:

$$R = (x^{\nu} - z^{\nu})\, u_{\nu}\,.$$

Because of the singular character of the potential $\{A^{\lambda}\}$ and the field $F^{\mu\nu} = A^{\mu/\nu} - A^{\nu/\mu}$, we have to exclude from our considerations a small region around the world line; however, the shape of the excluded region is very essential, because the large field strengths in the immediate neighbourhood of the singularity contribute most to the physically relevant quantities such as four-momentum or angular momentum of the charge. Therefore, it seems meaningful to us to start with some remarks about those excluded regions: the world tubes.

As was pointed out by Teitelboim [1] most clearly for the first time, the motivation for introducing such a tube was the fact that nobody

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was able to do the integration of the energy-momentum tensor $T_P{}^{\mu\nu}$

$$-4\pi\, T_P{}^{\mu\nu} = F^{\mu\lambda} F^{\nu}{}_{\lambda} - \tfrac{1}{4} g^{\mu\nu} (F^{\alpha\beta} F_{\alpha\beta})\,, \quad \text{(II,1)}$$

constructed with the retarded Liénard-Wiechert fields $F^{\mu\nu}$, over the orthogonal hyperplane $\sigma_\perp(s)$ in order to get the four-momentum $P_P{}^\mu$ of the retarded particle field:

$$P_P{}^\mu(s) = \frac{1}{c} \int_{\sigma_\perp(s)} T_P{}^{\mu\nu}\, \mathrm{d}^3\sigma_{\perp\nu}\,. \quad \text{(II,2)}$$

Also Teitelboim himself actually evaded the problem of a direct integration in (II,2) by resorting to an application of Gauß' integral theorem (see Fig. 1)

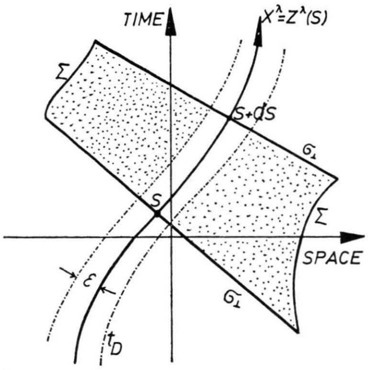

Fig. 1.

to the four-dimensional volume enclosed by two orthogonal hyperplanes at proper times s and $s+\mathrm{d}s$, a section of Dirac's tube (t_D) and by an infinitely distant surface (Σ), through which the flux of the retarded energy-momentum density vanishes, if one imposes the condition of uniform motion in the infinitely distant past:

$$\lim_{s\to-\infty} u^\lambda(s) = \text{const.} \quad \text{(II,3)}$$

Under these conditions, Gauß' integral theorem yields

$$\frac{\mathrm{d}P_P{}^\mu}{\mathrm{d}s} = -\frac{1}{c} \int_{(t_D)} T_P{}^{\mu\nu}\, \mathrm{d}^2 f_\nu\,. \quad \text{(II,4)}$$

Further, Teitelboim [1] observed that the whole energy-momentum density $T_P{}^{\mu\nu}$ can be split up into two parts (bound (b) and emitted (r) part)

$$T_P{}^{\mu\nu} = T_b{}^{\mu\nu} + T_r{}^{\mu\nu}\,, \quad \text{(II,5)}$$

each of which is conserved separately off the world line

$$T_b{}^{\mu\nu}{}_{/\nu} = T_r{}^{\mu\nu}{}_{/\nu} = 0\,, \quad \text{(II,6)}$$

so that one could define in (II,2) a bound four-momentum $\{P_b{}^\mu\}$ and an emitted four-momentum $\{P_r{}^\mu\}$ through

$$P_P{}^\mu = P_b{}^\mu + P_r{}^\mu \quad \text{(II,7a)}$$

$$P_b{}^\mu = \frac{1}{c} \int_{\sigma_\perp} T_b{}^{\mu\nu}\, \mathrm{d}^3\sigma_{\perp\nu} \quad \text{(II,7b)}$$

$$P_r{}^\mu = \frac{1}{c} \int_{\sigma_\perp} T_r{}^{\mu\nu}\, \mathrm{d}^3\sigma_{\perp\nu}\,. \quad \text{(II,7c)}$$

Correspondingly, the direct integrations (II,7b) and (II,7c) can again be evaded by resorting to (II,4)

$$\frac{\mathrm{d}P_b{}^\mu}{\mathrm{d}s} = -\frac{1}{c} \int T_b{}^{\mu\nu}\, \mathrm{d}^2 f_\nu \quad \text{(II,8a)}$$

$$= \frac{Z^2}{2c\,\varepsilon}\, \dot{u}^\mu - \frac{2}{3}\frac{Z^2}{c}\, \ddot{u}^\mu + O(\varepsilon)$$

$$\frac{\mathrm{d}P_r{}^\mu}{\mathrm{d}s} = -\frac{1}{c} \int T_r{}^{\mu\nu}\, \mathrm{d}^2 f_\nu = -\frac{2}{3}\frac{Z^2}{c} (\dot{u}\,\dot{u})\, u^\mu\,. \quad \text{(II,8b)}$$

With the asymptotic condition (II,3) the bound four-momentum (II,8a), which has the remarkable property to be a total differential, can be easily integrated to yield

$$P_b{}^\mu = \frac{Z^2}{2c\,\varepsilon}\, u^\mu - \frac{2}{3}\frac{Z^2}{c}\, \dot{u}^\mu\,, \quad \text{(II,9)}$$

which is a local quantity, i.e. it does not depend upon the earlier states of motion, though the corresponding density $T_b{}^{\mu\nu}$ on $\sigma_\perp(s)$ does depend upon the past world line.

In looking for an explanation of the local character of $P_b{}^\mu$, van Weert [9] invented a second way of evading the direct integration problem in form of (II,7a): He was able to show that the bound density $T_b{}^{\mu\nu}$ is the divergence of an anti-symmetric tensor of third rank ($K^{\mu\nu\lambda} = -K^{\mu\lambda\nu}$):

$$T_b{}^{\mu\nu} = K^{\mu\nu\lambda}{}_{/\lambda}\,, \quad \text{(II,10)}$$

so that one could apply Stokes' theorem in (II,7a) to obtain

$$P_b{}^\mu = \frac{1}{c} \iiint_{\Delta^3\sigma_\perp} K^{\mu\nu\lambda}{}_{/\lambda}\, \mathrm{d}^3\sigma_{\perp\nu} = -\frac{1}{2c} \iint_{\Delta^2\sigma_\perp} K^{\mu\nu\lambda}\, \mathrm{d}^2\sigma_{\nu\lambda}\,. \quad \text{(II,11)}$$

Here is $\Delta^2\sigma_\perp$ the two-dimensional boundary of the three-dimensional space $\Delta^3\sigma_\perp$ and consists of two closed two-surfaces in $\sigma_\perp$: a small one [$\Delta^2\sigma_\perp(\varepsilon)$; $\varepsilon\to 0$] in the immediate vicinity of the singularity and an infinitely distant one $[\Delta^2\sigma_\perp(\varrho);\ \varrho \to \infty]$. The latter one again depends in general upon the acceleration in the distant past; hence, one must also here impose the asymptotic condition (II,3) in order that the bound momentum is representable as an integral over the vicinity of the singularity and becomes thus a local quantity

$$P_{\mathrm{b}}{}^{\mu} = -\frac{1}{2c} \iint\limits_{\Delta^2\sigma_\perp(\varepsilon)} K^{\mu\nu\lambda}\,\mathrm{d}^2\sigma_{\nu\lambda}\,. \tag{II,12}$$

Here is $\mathrm{d}^2\sigma_{\nu\lambda}$ the oriented two-surface element of the small surface around the singularity

$$\mathrm{d}^2\sigma_{\nu\lambda} = \varepsilon_{\nu\lambda\varrho\mu}\,\mathrm{d}_1x^{\varrho}\,\mathrm{d}_2x^{\mu}\,, \tag{II,13}$$

where $\{\mathrm{d}_1x_{\varrho}\}$ and $\{\mathrm{d}_2x^{\mu}\}$ are any two independent variations on the two-surface $\Delta^2\sigma_\perp(\varepsilon)$. We want to point out here, that there are special two-surfaces $\Delta^2\sigma_\perp$, which lead automatically to the vanishing of the integral (II,11) over the outer surface $[\Delta^2\sigma_\perp(\varrho);\ \varrho \to \infty]$, so that one can dispense with the asymptotic condition (II,3) in this special case. This statement is proven explicitly in the appendix.

Before we come to the third way out of the trouble to integrate (II,2) directly, let us introduce a new tube, which exhibits in a very convenient form the instantaneous features of Dirac's tube with the retarded character of the electromagnetic fields [10]. This tube arises by intersecting the future light cone with vertex in $\hat{z}^{\lambda} := z^{\lambda}(s - \Delta s)$ on the world line and the orthogonal hyperplane $\sigma_\perp(s)$, which yields a closed two-surface $\Delta^2\bar{\sigma}_\perp$ in $\sigma_\perp$, and by varying proper time s under the constraint $\Delta s = \mathrm{const}$, the two-surface $\Delta^2\bar{\sigma}_\perp$ generates a tube $(\bar{t})$. The surface element $\mathrm{d}^3\bar{f}^{\nu}$ of this tube is

$$\begin{aligned}\mathrm{d}^2\bar{f}^{\nu} &= \mathrm{d}^3\bar{f}^{\nu}/\mathrm{d}s \\ &= \{-u^{\nu} + [1 + (\dot{u}\,\mathfrak{z}) - R(\hat{n}\,\dot{u})]\,\hat{n}^{\nu}\} \\ &\quad\cdot (R^2/(\hat{n}\,u))\,\mathrm{d}\hat{\Omega}\,. \end{aligned} \tag{II,14}$$

The primed quantities (like $\hat{n}$) refer to the event $\hat{z}^{\lambda} = z^{\lambda}(s - \Delta s)$ on the world line, whereas the unprimed ones refer to $z^{\lambda}(s)$. Furthermore,

$$\hat{n}^{\nu} = (\bar{x}^{\nu} - \hat{z}^{\nu})/R;\quad \{\bar{x}^{\nu}\} \in (\bar{t});\quad (\bar{x}^{\nu} - z^{\nu})\,u_{\nu} = 0\,, \tag{II,15}$$

$$\mathfrak{z}^{\lambda} := z^{\lambda} - \hat{z}^{\lambda}\,. \tag{II,16}$$

The use of the tube $(\bar{t})$ induces two new possibilities of calculating indirectly (II,2)! The first one is achieved by applying Gauß' integral theorem to the four-dimensional region bound by the light cone

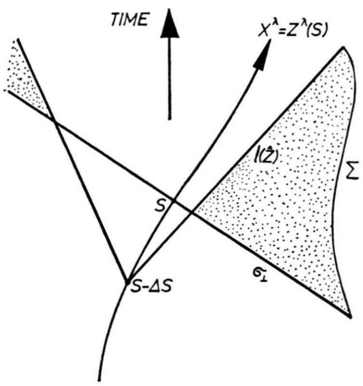

Fig. 2.

$l(\hat{z})$ with vertex in $\hat{z}^{\lambda}$ on the world line, by the hyperplane $\sigma_\perp(s)$, and by an infinitely distant surface (Σ) (see Fig. 2). Thus, one finds with the help of (II,2)

$$P_{\mathrm{P}}{}^{\mu} = \frac{1}{c}\int\limits_{l(\hat{z})} T_{\mathrm{P}}{}^{\mu\nu}\,\mathrm{d}^3\hat{l}_{\nu} - \frac{1}{c}\int\limits_{(\Sigma)} T_{\mathrm{P}}{}^{\mu\nu}\,\mathrm{d}^3\Sigma_{\nu}\,, \tag{II,17}$$

where the surface element of a light cone is given through [11]

$$\mathrm{d}^3\hat{l}^{\nu} = \hat{n}^{\nu}\,R^2\,\mathrm{d}R\,\mathrm{d}\hat{\Omega}\,. \tag{II,18}$$

Now it happens that the flux of the retarded density $T_{\mathrm{r}}{}^{\mu\nu}$ through the light cone $l(\hat{z})$ vanishes [1]

$$T_{\mathrm{r}}{}^{\mu\nu}\,\hat{n}_{\nu} \equiv 0\,, \tag{II,19}$$

so that one would be legitimated to redefine the bound four-momentum by a light cone integral

$$P_{\mathrm{b}}{}^{\mu}(s) = \frac{1}{c}\int\limits_{l(\hat{z})} P_{\mathrm{P}}{}^{\mu\nu}\,\mathrm{d}^3\hat{l}_{\nu} = \frac{1}{c}\int\limits_{l(\hat{z})} T_{\mathrm{b}}{}^{\mu\nu}\,\mathrm{d}^3\hat{l}_{\nu}\,, \tag{II,20}$$

if only the flux of $T_{\mathrm{b}}{}^{\mu\nu}$ through the infinitely distant surface (Σ) would vanish, because then one could apply Gauß' theorem to $T_{\mathrm{b}}{}^{\mu\nu}$ alone.

Under what condition vanishes the bound flux through the surface (Σ)?

This question is easily answered by evoking Stokes' theorem:

$$\iiint\limits_{(\Sigma)} T_{\mathrm{b}}{}^{\mu\nu}\,\mathrm{d}^3\Sigma_{\nu} = -\tfrac{1}{2}\iint\limits_{B(\Sigma)} K^{\mu\nu\lambda}\,\mathrm{d}^2\Sigma_{\nu\lambda}\,. \tag{II,21}$$

The boundary $B(\Sigma)$ of (Σ) consists of the intersection $B_{\sigma}(\Sigma)$ of (Σ) with $\sigma_\perp$ and of the intersection $B_{\mathrm{l}}(\Sigma)$ of (Σ) with $l(\hat{z})$. The intersection $B_{\sigma}(\Sigma)$ yields a contribution identical to the outer surface integration in van Weert's method (II,11) and therefore vanishes under the same condition as was discussed below (II,11). This condition is just the asymptotic requirement (II,3). As for the second boundary $B_1(\Sigma)$, we apply once more Stokes' Theorem to the light cone $l(\hat{z})$:

$$\iiint_{l(\hat{z})} T_b{}^{\mu\nu}\,d^3\hat{l}_\nu = \iiint_{l(\hat{z})} K^{\mu\nu\lambda}{}_{/\lambda}\,d^3\hat{l}_\nu = -\tfrac{1}{2}\iint_{B(l)} K^{\mu\nu\lambda}\,d^2\sigma_{\nu\lambda}\,. \qquad \text{(II,22)}$$

The boundary $B(l)$ on the light cone is composed of the intersection $B_1(\Sigma)$ of $l(\hat{z})$ and (Σ) and of the intersection $B_1(\sigma)$ of $l(\hat{z})$ and $\sigma_\perp(s)$. We can now determine both boundary integrals $B_1(\Sigma)$ and $B_1(\sigma)$ simultaneously by performing the R-integration on the left-hand side of (II,22); this procedure leaves the angular integration on $B_1(\Sigma)$ and $B_1(\sigma)$, which can thus be identified with the right-hand boundary integrals:

$$\frac{1}{2}\iint_{B_1(\Sigma)} K^{\mu\nu\lambda}\,d^2\sigma_{\nu\lambda} = \iint_{B_1(\Sigma)} d\hat{\Omega}\,\frac{Z^2}{8\pi R_m}\,\hat{n}^\mu \qquad \text{(II,23a)}$$

$$-\frac{1}{2}\iint_{B_1(\sigma)} K^{\mu\nu\lambda}\,d^2\sigma_{\nu\lambda} = \iint_{B_1(\sigma)} d\hat{\Omega}\,\frac{Z^2}{8\pi\varrho}\,(\hat{n}\,u)\,\hat{n}^\mu \qquad \text{(II,23b)}$$

where [10]

$$\varrho := (z^\nu - \hat{z}^\nu)\,u_\nu \qquad \text{(II,24)}$$

and R_m must be considered as a known function of the angles $(\hat{\Theta}, \hat{\Phi})$ with $d\hat{\Omega} = \sin\hat{\Theta}\,d\hat{\Theta}\,d\hat{\Phi}$. From (II,23a) we see that for an infinitely distant surface (Σ), which implies $R_m \to \infty$, the boundary integral over $B_1(\Sigma)$ vanishes, so that one concludes together with the vanishing of the $B_\sigma(\Sigma)$-integral that the integral (II,21) is really zero under the asymptotic condition (II,3). Hence, the redefinition of the bound four-momentum in (II,20) is justified and the result is [10]

$$P_b{}^\mu(s, \Delta s) = \frac{Z^2}{2c\,\varrho}\,u^\mu(s) - \frac{4}{3}\,\frac{Z^2}{2c\,\varrho}\,\{\hat{u}^\mu(u\,\hat{u}) - u^\mu\}\,. \qquad \text{(II,25)}$$

This formula shows very clearly that the naive expectation "four-momentum = mass times four-velocity" is no longer true in the present case but must be supplied by the second term on the right of (II,25). If the tube introduced with (II,14) shrinks to zero ($\varrho \to \Delta s \to 0$), this second term becomes just the "Schott term" $-\dfrac{2}{3}\dfrac{Z^2}{c}\,\dot{u}^\mu$. Hence, this term is interpretable as the relative four-momentum of the source with respect to the surrounding Coulomb cloud [12]. Because of the splitting (II,7a) and the identification (II,20), one concludes for the radiated four-momentum $P_r{}^\mu$ from (II,17)

$$P_r{}^\mu = \frac{1}{c}\int_{(\Sigma)} T_P{}^{\mu\nu}\,d^3\Sigma_\nu = \frac{1}{c}\int_{(\Sigma)} T_r{}^{\mu\nu}\,d^3\Sigma_\nu\,. \qquad \text{(II,26)}$$

The result of this integration is independent of the special choice of (Σ) and coincides with (II,8b):

$$P_r{}^\mu(s) = -\frac{2}{3}\,\frac{Z^2}{c}\int_{-\infty}^{s} (\dot{u}\,\dot{u})\,u^\mu(s')\,ds'\,. \qquad \text{(II,27)}$$

It is now easy to see that the asymptotic condition (II,3) is not needed, if we use the tube $(\bar{t})$ as the infinitely distant surface (Σ) where Δs must tend to infinity, because the boundary $B_\sigma(\Sigma)$ occuring in (II,21) gives the same contribution as in van Weert's method if applied to a light-cone intersection boundary (see Appendix). The same reasoning holds also for Teitelboim's original procedure, for one can convert the integral over Teitelboim's (Σ) (Fig. 1) in a boundary integral (Stokes' Theorem) over the intersection of (Σ) and of the orthogonal planes $\sigma_\perp$. These intersections are light cone intersections if tube $(\bar{t})$ is used to generate (Σ). But then the result of the appendix applies, and one can also in Teitelboim's method dispense with the uniform motion condition.

Finally, we want to report a final possibility of indirect calculating $P_P{}^\mu$, which resembles most the direct integration method to be represented in the next section. One applies Gauß integral theorem to the four-dimensional volume bounded by $\sigma_\perp(s)$, the tube $(\bar{t})$, and the light cone l_∞ with vertex in the distant past (see Fig. 3)

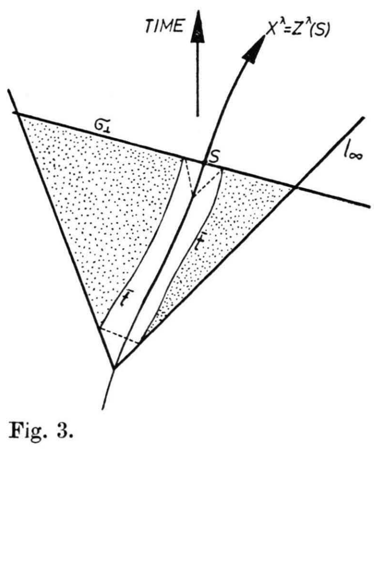

Fig. 3.

$$c\,P_{\rm P}{}^{\mu}(s) = \int_{\sigma_\perp(s)} T_{\rm P}{}^{\mu\nu}\,{\rm d}^3\sigma_{\perp\nu} = -\int_{-\infty(\bar{t})}^{s} T_{\rm P}{}^{\mu\nu}\,{\rm d}^3\bar{f}_\nu + \int_{l_\infty} T_{\rm P}{}^{\mu\nu}\,{\rm d}^3 l_\nu\,. \qquad \text{(II,28)}$$

Here, the right-hand side is easily calculated by using the surface element (II,14) and the light-cone element (II,18)

$$-\frac{1}{c}\int_{-\infty}^{s} T_{\rm P}{}^{\mu\nu}\,{\rm d}^3\bar{f}_\nu = P_{\rm b}{}^{\mu}(s,\Delta s) - P_{\rm b}{}^{\mu}(-\infty,\Delta s) + P_{\rm r}{}^{\mu}(s)\,, \qquad \text{(II,29)}$$

where $P_{\rm b}{}^{\mu}$ is already given by (II,25) and $P_{\rm r}{}^{\mu}$ by (II,27). The light-cone integral on the right of (II,28) is calculated as

$$\int_{l_\infty} T_{\rm P}{}^{\mu\nu}\,{\rm d}^3 l_\nu = \int_{l_\infty} T_{\rm b}{}^{\mu\nu}\,{\rm d}^3 l_\nu = P_{\rm b}{}^{\mu}(-\infty,\Delta s) - P_{\rm b}{}^{\mu}[s, s'\to-\infty]\,. \qquad \text{(II,30)}$$

Here, we have put

$$P_{\rm b}{}^{\mu}[s,s'] := \frac{Z^2}{2c\,\varrho(s,s')}\left\{\frac{4}{3}\,u'^{\mu}(u\,u') - \frac{1}{3}\,u^{\mu}\right\}.$$

The contribution $P_{\rm b}{}^{\mu}[s, s'\to-\infty]$ can be dropped under certain conditions, which are discussed more thoroughly in the next section, where the nature of $P_{\rm b}{}^{\mu}[s,s']$ becomes evident in the course of the direct integration.

Inserting (II,29) and (II,30) into (II,28), one finds of course the old expressions for $P_{\rm P}{}^{\mu}$. However, according to the use of our tube ($\bar{t}$), we were not forced to make use of the asymptotic condition (II,3) with respect to the bound part of the four-momentum. Would we have used Dirac's tube, then there would have arisen a certain "matching problem" of the surface, which could have resolved only by the introduction of the asymptotic condition of uniform motion in the distant past [13]. We see here that the asymptotic condition is not needed whenever we integrate over $\sigma_\perp$ in such a way that the outer boundary to be shifted to infinity is taken to be the intersection of a light cone with vertex in some event on the past world line and of the hyperplane $\sigma_\perp$. This phenomenon occurs in all the evading methods described in this section, but the direct integration method of the next section shall bring out this effect more clearly.

III. Retarded Integration

In contradistinction to all the previous methods, we want to perform now the integration in the defining equation (II,2) directly. For the sake of convenience, we compute the bound and irradiated part of the particle four-momentum $P_{\rm P}{}^{\mu}$ separately, though this is actually not necessary; the Teitelboim splitting is nowhere needed for the present method.

The essential point is that we integrate first over the region of $\sigma_\perp$, which is hit by the electromagnetic excitation originating in a single event on the past world line, say $z^\lambda(s')$; $s'<s$ (see Figure 4). This

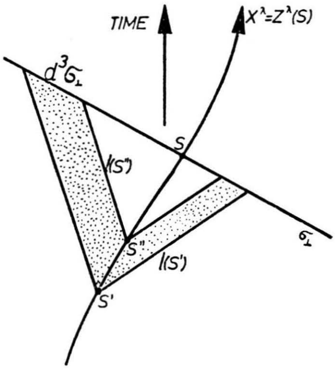

Fig. 4.

region is the intersection of the light cone $l(s')$ with vertex in $z^\lambda(s')$ and of the plane $\sigma_\perp(s)$. Therefore, we put

$${\rm d}^3\sigma_\perp{}^{\nu} = {\rm d}^2\sigma_\perp{}^{\nu}\cdot{\rm d}s' \qquad \text{(III,1)}$$

and express ${\rm d}^2\sigma_\perp{}^{\nu}$ by its orthogonal projection on the orthogonal hyperplane $\sigma_\perp(s')$ in the past event $z^\lambda(s')$. The result is [14]

$${\rm d}^3\sigma_\perp{}^{\nu} = u^\nu\,\frac{R'^2}{(n'\,u)}\,{\rm d}\Omega'\,{\rm d}s'\,. \qquad \text{(III,2)}$$

For the convenience of the reader, a short verification of the surface element can be given in the following, but one should observe that this plausibility consideration has nothing to do with the actual integration (II,2).

Applying Gauß' theorem to the four-volume bounded by the light-cone $l(s')$, $l(s'+{\rm d}s')$, and $\sigma_\perp(s)$ (see Fig. 4) yields

$$0 = \int {\rm d}^4x\, g^{\mu\nu}{}_{/\nu} = \oint g^{\mu\nu}\,{\rm d}^3\sigma_\nu = \oint {\rm d}^3\sigma^\mu = \int_{l(s'+{\rm d}s')} {\rm d}^3 l^\mu - \int_{l(s')} {\rm d}^3 l^\mu + \int {\rm d}^3\sigma_\perp{}^{\mu}\,. \qquad \text{(III,3)}$$

However, a light-cone integral is easily evaluated as

$$\int_{l(s')} {\rm d}^3 l^\mu = \iint {\rm d}\Omega'\,n'^\mu \int_0^{R_{\rm m}'} R'^2\,{\rm d}R'\,, \qquad \text{(III,4)}$$

where

$$R_m' = \varrho'/(n'u); \quad \varrho' = (z^\nu(s) - z^\nu(s'))\, u_\nu(s)\,. \tag{III,5}$$

The result of integration is

$$\int_{l(s')} d^3 l^\mu = \tfrac{1}{3}\varrho'^3 \iint d\Omega' \,[n'^\mu/(n'u)^3] = \frac{4\pi}{3}\varrho'^3 u^\mu\,, \tag{III,6}$$

and inserting this into (III,3) yields

$$\int d^3\sigma_\perp{}^\mu = \frac{4\pi}{3} u^\mu \{\varrho^3(s') - \varrho^3(s' + ds')\} \tag{III,7}$$
$$= -4\pi u^\mu \varrho'^2 \frac{\partial \varrho'}{\partial s'} ds' = 4\pi u^\mu \varrho'^2 (u\,u')\, ds'\,.$$

Observing the angular integration

$$\int [d\Omega'/(n'u)^3] = 4\pi (u\,u')\,, \tag{III,8}$$

we finally have

$$\int d^3\sigma_\perp{}^\mu = \int d\Omega'\, ds' \,[R'^2/(n'u)]\, u^\mu\,, \tag{III,9}$$

which readily implies (III,2) for the integrands.

For the actual integration, we need the explicit expression for the energy-momentum densities

$$-4\pi T_b{}^{\mu\nu} = (Z^2/R'^4)\{n'^\mu n'^\nu - u'^\mu n'^\nu - u'^\nu n'^\mu + \tfrac{1}{2} g^{\mu\nu}\} + (Z^2/R'^3)\{n'\dot u')[u'^\mu n'^\nu + u'^\nu n'^\mu - n'^\mu n'^\nu] - n'^\mu \dot u'^\nu - n'^\nu \dot u'^\mu\} \tag{III,10a}$$

$$-4\pi T_r{}^{\mu\nu} = (Z^2/R'^2)\{(n'\dot u')^2 + (\dot u'\dot u')\} n'^\mu n'^\nu\,. \tag{III,10b}$$

Computing first the emitted part $P_r{}^\mu$, we conclude from (III,2) and (III,10b)

$$P_r{}^\mu = \frac{1}{c}\int d^3\sigma_{\perp\nu}\, T_r{}^{\mu\nu} \tag{III,11}$$
$$= -\frac{Z^2}{4\pi c}\int_{-\infty}^{s} d\Omega'\, ds' \{(n'\dot u')^2 + (\dot u'\dot u')\}\, n'^\mu$$
$$= -\frac{2}{3}\frac{Z^2}{c}\int_{-\infty}^{s} (\dot u'\dot u')\, u'^\mu\, ds'\,.$$

If we look at the integrand of (III,11), we see that the normal $\{u^\nu\}$ of the plane $\sigma_\perp(s)$ has cancelled, so that the result is independent of the direction of the hyperplane. This independence is a well-known feature of the radiated momentum (cf. the previous methods, where $P_r{}^\mu$ could be obtained by an integral over an arbitrary surface (Σ)).

In order to compute the bound four-momentum $P_b{}^\mu$ by means of this direct method, we proceed in two steps by writing

$$P_b{}^\mu = P^\mu_{(-3)} + P^\mu_{(-4)}\,, \tag{III,12a}$$

$$P^\mu_{(-3)} = \frac{1}{c}\int_{\sigma_\perp} T^{\mu\nu}_{(-3)}\, d^3\sigma_{\perp\nu}\,, \tag{III,12b}$$

$$P^\mu_{(-4)} = \frac{1}{c}\int_{\sigma_\perp} T^{\mu\nu}_{(-4)}\, d^3\sigma_{\perp\nu}\,, \tag{III,12c}$$

where $T^{\mu\nu}_{(-3)}$ and $T^{\mu\nu}_{(-4)}$ are the contributions to $T_b{}^{\mu\nu}$, which are proportional to R'^{-3} and R'^{-4}, respectively. Performing now all the angular integrations involved in (III,12b) and (III,12c) leads to

$$P^\mu_{(-3)} = \frac{2}{3}\frac{Z^2}{c}\int_{-\infty}^{s} ds' \frac{1}{\varrho'}\{\dot u'^\mu (u\,u') + (u\,\dot u')\, u'^\mu\} \tag{III,13a}$$

$$P^\mu_{(-4)} = \frac{Z^2}{2c}\int_{-\infty}^{s} ds' \frac{(u\,u')}{\varrho'^2}\left\{\frac{4}{3} u'^\mu (u\,u') - \frac{1}{3} u^\mu\right\}. \tag{III,13b}$$

These two contributions to $P_b{}^\mu$ add up to give

$$P_b{}^\mu = \frac{Z^2}{c}\int_{-\infty}^{s} ds' \frac{\partial}{\partial s'}\left\{\frac{1}{2\varrho'}\left[\frac{4}{3}(u\,u')\, u'^\mu - \frac{1}{3} u^\mu\right]\right\}, \tag{III,14}$$

where the integrand is obviously a total derivative with respect to the variable s'. We can therefore perform the s'-integration to obtain

$$P_b{}^\mu(s) = \left(\lim_{s'\to s} - \lim_{s'\to -\infty}\right) \cdot \left\{\frac{Z^2}{2c\varrho'}\left[\frac{4}{3}(u\,u')\, u'^\mu - \frac{1}{3} u^\mu\right]\right\}. \tag{III,15}$$

For $s' \to s$, one observes the usual Coulomb divergence plus the Schott term

$$\frac{Z^2}{2c(s-s')}\dot u^\mu(s) - \frac{2}{3}\frac{Z^2}{c}\dot u^\mu(s)\,,$$

whereas for $s' \to -\infty$ the contribution vanishes, if $u'^\lambda/\varrho' \to 0$, i.e.

$$\lim_{s'\to-\infty} \frac{u^\lambda(s')}{(z^\nu(s) - z^\nu(s'))\, u_\nu(s)} = 0\,. \tag{III,16}$$

This is a condition upon $u^\lambda(s \to -\infty)$, but it is much weaker than the asymptotic condition (II,3) of uniform motion in the distant past. For instance, every motion with bounded four-velocity in the distant past will lead to a vanishing contribution to $P_b{}^\mu$ in (III,15) for $s' \to -\infty$.

So we see that choosing the "physical boundary" (= intersection of light cone $l(s' \to -\infty)$ and hyperplane $\sigma_\perp(s)$ has led to the possibility again to dispense with the asymptotic condition (II,3) just as was the case for all the indirect methods reported in the foregoing section. However, whereas one had to prescribe the "physical boundary" in an artificial manner for the indirect methods, the present direct integration method automatically works with the „physical boundary".

IV. Some Arguments in Favour of the "Physical Boundary"

After having elaborated the implications which allow us to renounce the asymptotic condition of uniform motion in the distant past, we want now give some arguments in favour of the choice of the "physical boundary" in performing the integration (II,2).

Consider the world line of Fig. 4 and suppose that we switch-on the electromagnetic interaction only between proper times s' and $s'' = s' + ds'$ by creating the charge Z in the event s' and destroying it in s'', so that the particle is uncharged during the periods $s < s'$ and $s > s''$. In order to calculate the four-momentum emitted by the charge in proper time interval $[s', s'']$, we have to choose an arbitrary surface (Σ), which is cut by the light cones $l(s')$ and $l(s'')$. The integration (II,7c) leading to $P_r{}^\mu$ must be performed over that surface (Σ) spanned between the two light cones $l(s')$ and $l(s'')$. If we would integrate over an arbitrary surface (Σ'), which is not bounded by the two light cones (such as proposed by Tabensky [15], see Fig. 5), we would not get a physical quantity, which is uniquely related to the section $[s', s'']$ of the world line, because the radiation between events A, A' and B, B' has been suppressed for some spatial directions (dotted lines in Fig. 5). The result of such an integration would depend strongly upon the chosen surface (Σ') and would not reflect the behaviour of the charge in $[s', s'']$ in a unique way. Only if the physical boundary determined by the two light cones $l(s')$ and $l(s'')$ are used, we can attach a physically meaningful quantity

$$P_r{}^\mu[s', s''] = -\frac{2}{3}\frac{Z^2}{c}\int_{s'}^{s''}(\dot{u}^*\,\dot{u}^*)\,u^\mu(s^*)\,ds^*$$

to the world line section $[s', s'']$.

Transferring this way of reasoning to the calculation of the bound four-momentum $P_b{}^\mu$ as

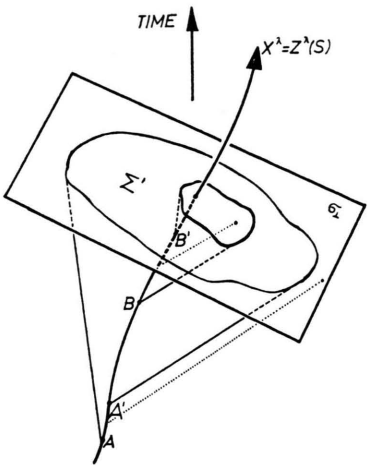

Fig. 5.

given in (II,7b), we see that one must first perform the integration with finite boundary, which has to be built from light cone sections, and then the vertices of these light cones may be shifted to the present event $(s'' \to s)$ or to the infinite past $(s' \to -\infty)$. The contribution $P_b{}^\mu[s', s'']$ of the finite section $[s', s'']$ to the bound four-momentum $P_b{}^\mu(s)$ is then a unique functional of the world line section $[s', s'']$ and consequently $P_b{}^\mu(s)$ is a unique functional of the whole past history of the electron (albeit it turns out to depend only upon the present event)). Otherwise, using not the physical boundary, $P_b{}^\mu(s)$ would depend upon the acceleration in the distant past and therefore would be a non-local quantity. In order to remedy the mistake of not having chosen the „physical boundary" one must impose the asymptotic condition (II,3), because then the result becomes identical to that one obtained by use of the "physical boundary".

So we see that we are forced to use the "physical boundary" in order to get physical quantities attached in a unique maner to arbitrary world line sections. But in doing so we can define the bound (and in some sense also the emitted) four-momentum without use of the restricting condition of uniform motion in the distant past. Instead, a much weaker condition holds, as was discussed at the end of the foregoing section.

Appendix

The „physical boundary" $\Delta^2\overline{\sigma_\perp}$ is the intersection of a light cone $l(s')$ and $\sigma_\perp(s)$. Obviously, the contribution $P_b{}^\mu[s', s'']$ of a past world line section (see Fig. 4, where $s' + ds' = s''$) to the bound four-momentum $P_b{}^\mu$ is made up of integrals over such two-surfaces $\Delta^2 \overline{\sigma_\perp}$, each of which is uniquely related to one single source-point on the past world line. Integrating over these special two-surfaces in $\sigma_\perp(s)$ and then summing up all two-surfaces evidently means the summation over all past events on the world line in their chronological order. Due to the special structure (II,10) of the bound energy momentum density, only the contribution of the first and last two-surface, corresponding to the boundary of the interval $[s', s'']$, survives the summation process.

For an actual calculation, we first look for the two-surface element $d^2\sigma_{\nu\lambda}$ of an arbitrary "physical" two-surface. Its general definition is

$$d^2\sigma_{\nu\lambda} = \varepsilon_{\nu\lambda\varrho\gamma} \frac{\partial \xi^\varrho}{\partial \Theta'} \cdot \frac{\partial \xi^\gamma}{\partial \Phi'} \, d\Theta' \, d\Phi' , \tag{A.1}$$

where $\{\xi^\lambda\}$ is some point on the two-surface $\Delta^2 \overline{\sigma_\perp}$:

$$\xi^\lambda = z'^\lambda + \varrho' \, [n'^\lambda/(n' u)] , \tag{A.2}$$

$$z'^\lambda = z^\lambda(s') , \tag{A.3}$$

$$\varrho' = \varrho(s, s') = (z^\nu(s) - z^\nu(s')) \, u_\nu(s) . \tag{A.4}$$

The null vector n'^λ

$$n'^\lambda := (\xi^\lambda - z'^\lambda)/R'; \quad R' = \varrho'/(n' u) \tag{A.5}$$

is decomposed as

$$n'^\lambda = u'^\lambda + v'^\lambda , \tag{A.6}$$

and $\{v'^\lambda\}$ is the space-like unit vector in $\sigma_\perp(s')$. This unit vector points in "radial" direction and can be supplemented by two mutually orthogonal vectors, $v_\Theta'^\lambda$ and $v_\Phi'^\lambda$, to an orthonormal triad in $\sigma_\perp(s')$:

$$\partial \xi^\lambda / \partial \Theta' = R' \, v_\Theta'^\lambda - R' \, (v_\Theta'^\lambda \, u) \frac{n'^\lambda}{(n' u)} \tag{A.7}$$

$$\partial \xi^\lambda / \partial \Phi' = R' \, v_\Phi'^\lambda \sin \Theta' . \tag{A.8}$$

Inserting this in (A.1) yields

$$d^2\sigma_{\nu\lambda} = [(n'_\nu u'_\lambda - n'_\lambda u'_\nu) - ((u \, v_\Theta')/(n' u)) \cdot (n'_\nu v_{\Theta\lambda}' - n'_\lambda v_{\Theta\nu}')] \, R'^2 \, d\Omega' . \tag{A.9}$$

Next we have to specify the third-rank tensor $K^{\mu\nu\lambda}$ on the hyperplane $\sigma_\perp(s)$:

$$- \frac{4\pi}{Z^2} K^{\mu\nu\lambda} = \frac{n'^\mu}{R'^2} \{(\dot{u}'^\nu n'^\lambda - \dot{u}'^\lambda n'^\nu) - (n' \, \dot{u}')(u'^\nu n'^\lambda - u'^\lambda n'^\nu)\} - \frac{1}{4} \frac{1}{R'^3} \{(g^{\mu\nu} n'^\lambda - g^{\mu\lambda} n'^\nu) + 3 n'^\mu (n'^\nu u'^\lambda - n'^\lambda u'^\nu)\} . \tag{A.10}$$

If we now calculate the contraction $K^{\mu\nu\lambda} d^2\sigma_{\nu\lambda}$, all the R'-independent terms cancel, and one finds the very simple result

$$- \frac{4\pi}{Z^2} K^{\mu\nu\lambda} d^2\sigma_{\nu\lambda} = \frac{n'^\mu}{R'} d\Omega' , \tag{A.11}$$

where $d\Omega' = \sin \Theta' \, d\Phi' \, d\Theta'$. Since there are no R'independent terms present, we are not forced to impose the condition of vanishing acceleration in the distant past in order to get rid of the R'-independent contribution from the outer boundary. The solid angle integration in (A.11) is easily performed to obtain from (II,12) the contribution $P_b{}^\mu[s', s'']$ of the world line section $[s', s'']$

$$P_b{}^\mu[s', s''] = (Z^2/2c\,\varrho'') \{\tfrac{4}{3} u''^\mu (u \, u'') - \tfrac{1}{3} u^\mu\} - (Z^2/2c\,\varrho') \{\tfrac{4}{3} u'^\mu (u \, u') - \tfrac{1}{3} u^\mu\} , \tag{A.12}$$

where

$$\varrho' = \varrho(s, s') = (z^\lambda(s) - z^\lambda(s')) \, u_\lambda(s) \to \infty \quad \text{for} \quad s' \to -\infty ,$$

$$\varrho'' = \varrho(s, s'') = (z^\lambda(s) - z^\lambda(s'') \, u_\lambda(s) \to s - s'' \quad \text{for} \quad s'' \to s .$$

If we want to have the contribution of the whole past world line, we let s' tend to $-\infty$ and s'' tend to s, where the second term of the right of (A.12) vanishes under conditions discussed at the end of Sect. III, and the first gives the divergent Coulomb term plus the Schott term (cf. (III,15)).

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